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AN ELASTIC PLASTIC NON-STEADY IDEAL FLOW SOLUTION AND ITS APPLICATION TO BENDING UNDER TENSION

Elena Lyamina, Sergei Alexandrov
Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow, 119526, Russia

ABSTRACT

A new analytic solution for plane-strain bending under tension at large strains is proposed. The solution is based on the ideal flow theory for elastic perfectly plastic incompressible materials. The variation of the bending moment and tensile force with process parameters is found and discussed.

Keywords: *Bending under tension, analytic solution, ideal flow, elasto-plasticity.*

1. INTRODUCTION

Ideal plastic flows constitute a class of solutions in the classical theory of plasticity [1]. They are defined by the condition that all material elements follow the minimum plastic work path, a condition which is believed to be advantageous for forming processes. The available ideal flow solutions are based on rigid plastic material models. Nevertheless, it has been shown in [2] that it is possible to satisfy the ideal flow conditions in steady flows of elastic plastic solids. The present paper provides an analytic non-steady plane strain ideal flow solution for elastic perfectly plastic material. The solution is suitable for describing the process of bending under tension. Such an idealization is often adopted for studying practical problems in sheet metal forming [3, 4]. The first analytic solution for plane strain bending under tension at large strains has been found in [5] for rigid perfectly plastic material. An ideal flow solution for several rigid plastic models has been provided in [6]. The present solution is based on the extension of the general method for analysis of pure plane strain bending developed in [7] and thus is restricted to incompressible materials.

2. GENERAL SOLUTION

A general approach to analysis of plane-strain pure bending of a sheet of incompressible elastic and rigid plastic material has been proposed in [7]. It is possible to verify by inspection that the part of this solution related to kinematics can be used to describe plane-strain bending under tension without any modification. In particular, the mapping between Eulerian Cartesian coordinates xy and Lagrangian coordinates $\xi\eta$ is given by

$$x/H = \sqrt{\zeta/a + s/a^2}\cos(2a\eta) - \sqrt{s}/a, \quad y/H = \sqrt{\zeta/a + s/a^2}\sin(2a\eta) \qquad (1)$$

where H is the initial thickness of the sheet, a is a function of the time, t, satisfying the condition a=0 at t=0 and s is a function of a which should be found by means of stress boundary conditions. In addition, the function $s(a)$ must satisfy the condition

$$s = 1/4 \quad \text{at} \quad a = 0 \qquad (2)$$

This condition ensures that $x = \zeta H$ and $y = \eta H$ at the initial instant. It is possible to assume with no loss of generality that the initial shape of the sheet is defined by the equations $x = -H$, $x = 0$ and $y = \pm L$ where $2L$ is the width of the sheet. Then, the shape of the sheet after any amount of deformation is determined by the equation $\zeta = -1$, $\zeta = 0$ and $\eta = \pm L/H$. Because of symmetry it is sufficient to consider the domain $\eta \geq 0$ (or $y \geq 0$). It is also convenient to introduce a moving cylindrical coordinate system $r\theta$ by the following transformation equations [7].

$$r/H = \sqrt{\zeta/a + s/a^2}, \qquad \theta = 2a\eta \qquad (3)$$

The origin of this coordinate system is located at $x/H = -\sqrt{s}/a$ and $y = 0$. Equations (3) give

$$r_{ou}/H = \sqrt{s}/a, \quad r_{in}/H = \sqrt{s/a^2 - 1/a}, \quad \theta_0 = 2aL/H, \quad h/H = \left(\sqrt{s} - \sqrt{s-a}\right)/a \qquad (4)$$

where r_{ou} is the outer radius of the sheet, r_{in} is the inner radius of the sheet, θ_0 is the orientation of the edge of the sheet, and h is the current thickness of the sheet (Fig.1). Using (1) the principal logarithmic strains are determined as

$$\varepsilon_{\zeta\zeta} = -\varepsilon_{\eta\eta} = -(1/2)\ln\left[4(\zeta a + s)\right] \qquad (5)$$

The position of the neutral line is given by

$$\zeta = \zeta_0 = -ds/da \qquad (6)$$

where ζ_0 is in general a function of *a*. The equilibrium equations reduce to

$$\frac{\partial \sigma_{\zeta\zeta}}{\partial \zeta} = \frac{a\left(s_{\eta\eta} - s_{\zeta\zeta}\right)}{2\left(\zeta a + s\right)} \tag{7}$$

where $\sigma_{\zeta\zeta}$ is the stress component in the Lagrangian coordinates ($\sigma_{\eta\eta}$ will stand for the other stress component), $s_{\zeta\zeta}$ and $s_{\eta\eta}$ are the deviatoric stress components. The difference $s_{\eta\eta} - s_{\zeta\zeta}$ can be found from the constitutive equation for this or that material model. It also follows from (3) that $\sigma_{\zeta\zeta} \equiv \sigma_{rr}$ and $\sigma_{\eta\eta} \equiv \sigma_{\theta\theta}$. The solution to (7) along with the boundary conditions on $\sigma_{\zeta\zeta}$ at $r = r_{in}$ (or $\zeta = -1$) and $r = r_{ou}$ (or $\zeta = 0$) determines and thus the complete solution to the problem. In particular, the solution in [7] has been obtained for $\sigma_{\zeta\zeta} = 0$ at $r = r_{in}$ and $r = r_{ou}$.

3. IDEAL FLOW SOLUTION

An ideal flow solution is obtained if the neutral line is fixed in the material. Therefore, it is necessary to put $\zeta_0 = constant$ in (6). Then, the solution to this equation satisfying (2) is

$$s = -\zeta_0 a + 1/4 \tag{8}$$

In the case of incompressible elastic perfectly plastic material the constitutive equations give, with the use of (5),

$$s_{\zeta\zeta} = -G\ln\left[4\left(\zeta a + s\right)\right], \quad s_{\eta\eta} = G\ln\left[4\left(\zeta a + s\right)\right] \tag{9}$$

In the elastic zone and

$$\left|s_{\zeta\zeta}\right| = \left|s_{\eta\eta}\right| = \sigma_0/\sqrt{3} \tag{10}$$

in the plastic zone. Here σ_0 is the yield stress in tension, a material constant, and *G* is the shear modulus.

3.1 Elastic Stage

Substituting (9) into (7), integrating with the boundary condition $\sigma_{\zeta\zeta} = 0$ at $\zeta = 0$ and excluding *s* by means of (8) gives

$$2\sigma_{rr}/G = \ln^2\left[4a\left(\zeta - \zeta_0\right) + 1\right] - \ln^2\left(1 - 4a\zeta_0\right) \tag{11}$$

Using the identity $\sigma_{rr} - \sigma_{\theta\theta} \equiv s_{rr} - s_{\theta\theta}$ and (11) leads to

$$2\sigma_{\theta\theta}/G = \ln^2\left[4a\left(\zeta - \zeta_0\right) + 1\right] - \ln^2\left(1 - 4a\zeta_0\right) + 4\ln\left[4a\left(\zeta - \zeta_0\right) + 1\right] \tag{12}$$

It follows from (8), (9) and (10) that this stage ends when at least one of the following conditions is satisfied

$$k = -\ln\left[1 - 4a\left(\zeta_0 + 1\right)\right], \quad k = \ln\left(1 - 4\zeta_0 a\right), \quad k = \sigma_0 \Big/ \left(\sqrt{3}G\right) \tag{13}$$

These relations show that the plastic zones at $r = r_{in}$ and $r = r_{ou}$ starts to develop simultaneously if

$$\zeta_0 = \zeta_{cr} = \left(1 - e^k\right) \Big/ \left(2\sinh k\right) \tag{14}$$

Since $k > 0$, it follows from (14) that $\zeta_{cr} < -1/2$. Moreover, the plastic zone first appears at $r = r_{in}$ if $\zeta_0 > \zeta_{cr}$ and, consequently, at $r = r_{ou}$ if $\zeta_0 < \zeta_{cr}$.

3.2 Elastic Plastic Stage with One Plastic Zone

Assume that $\zeta_0 < \zeta_{cr}$. Then, the plastic zone starts to develop from the boundary $r = r_{ou}$ at

$$a = a_2 = \left(1 - e^k\right) \Big/ \left(4\zeta_0\right) \tag{15}$$

There are two zones at $a > a_2$, the elastic zone in the domain $-1 \le \zeta \le \zeta_2$ and the plastic zone in the domain $\zeta_2 \le \zeta \le 0$. Equation (10) in the plastic zone reduces to $-s_{\zeta\zeta} = s_{\eta\eta} = \sigma_0 \big/ \sqrt{3}$. Therefore, equation (7) becomes

$$\frac{\partial \sigma_{\zeta\zeta}}{\partial \zeta} = \frac{\sigma_0 a}{\sqrt{3}\left(\zeta a + s\right)} \tag{16}$$

and the elastic plastic boundary is determined with the use of (9) in the form

$$\zeta_2 = \left(e^k - 4s\right) \Big/ \left(4a\right) \tag{17}$$

The distribution of the stress $\sigma_{\zeta\zeta}$ in the plastic zone follows from the solution to equation (16) satisfying the boundary condition $\sigma_{\zeta\zeta} = 0$ for $\zeta = 0$

$$\frac{\sigma_{\zeta\zeta}}{\sigma_0} = \frac{\sigma_{rr}}{\sigma_0} = \frac{1}{\sqrt{3}} \ln\left[\frac{\zeta a + s}{s}\right] \tag{18}$$

The magnitude of $\sigma_{\zeta\zeta}$ on the plastic side of the elastic plastic boundary can be found from (18) at $\zeta = \zeta_2$. Substituting (9) into (7) and integrating with the condition of continuity of $\sigma_{\zeta\zeta}$ across the elastic plastic boundary gives the distribution of the stress $\sigma_{\zeta\zeta}$ in the elastic zone

$$\frac{\sigma_{\zeta\zeta}}{\sigma_0} = \frac{\sigma_{rr}}{\sigma_0} = \frac{1}{2\sqrt{3}k}\left\{\ln^2\left[4\left(a\zeta + s\right)\right] - \ln^2\left[4\left(a\zeta_2 + s\right)\right]\right\} + \frac{1}{\sqrt{3}}\ln\left(\frac{a\zeta_2 + s}{s}\right) \tag{19}$$

Using the identity $\sigma_{rr} - \sigma_{\theta\theta} \equiv s_{rr} - s_{\theta\theta}$, (9), (10), (18) and (19) leads to

$$\frac{\sigma_{\theta\theta}}{\sigma_0} = \frac{1}{2\sqrt{3}k}\left\{\ln^2\left[4\left(a\zeta + s\right)\right] - \ln^2\left[4\left(a\zeta_2 + s\right)\right]\right\} + \frac{1}{\sqrt{3}}\ln\left(\frac{a\zeta_2 + s}{s}\right) + \frac{2}{\sqrt{3}k}\ln\left[4\left(a\zeta + s\right)\right] \tag{20}$$

in the elastic zone and

$$\frac{\sigma_{\theta\theta}}{\sigma_0} = \frac{1}{\sqrt{3}}\ln\left[\frac{\zeta a + s}{s}\right] + \frac{2}{\sqrt{3}} \tag{21}$$

in the plastic zone. The pressure over the surface $r = r_{in}$ is determined from (19) at $\zeta = -1$

$$\frac{p}{\sigma_0} = -\frac{\sigma_{rr}\big|_{\zeta=-1}}{\sigma_0} = -\frac{1}{2\sqrt{3}k}\left\{\ln^2\left[4\left(s - a\right)\right] - \ln^2\left[4\left(a\zeta_2 + s\right)\right]\right\} - \frac{1}{\sqrt{3}}\ln\left(\frac{a\zeta_2 + s}{s}\right) \tag{22}$$

This stage ends when the plastic zone starts to develop at $\zeta = -1$. The corresponding value of a can be found from the first of equations (13) in the form

$$a = a_1 = \left(1 - e^{-k}\right) \Big/ \left[4\left(\zeta_0 + 1\right)\right] \tag{23}$$

Equations (8) and (17) should be used to exclude s and ζ_2 everywhere in this section.
A similar analysis can be completed if $\zeta_0 > \zeta_{cr}$. In this case however the pressure over the surface $r = r_{in}$ is negative. Therefore, this case is not considered here.

3.3 Elastic Plastic Stage with Two Plastic Zones

There are three zones at $a > a_1$, the elastic zone in the domain $\zeta_2 \geq \zeta \geq \zeta_1$ and the plastic zones in the domains $\zeta_1 \geq \zeta \geq -1$ and $0 \geq \zeta \geq \zeta_2$. The solution given by (18) and (21) is valid in the plastic zone $0 \geq \zeta \geq \zeta_2$. The solution given by (19) and (20) is valid in the elastic zone. Equation (10) in the plastic zone $\zeta_1 \geq \zeta \geq -1$ reduces to $s_{\zeta\zeta} = -s_{\eta\eta} = \sigma_0 / \sqrt{3}$. Therefore, equation (7) becomes

$$\frac{\partial \sigma_{\zeta\zeta}}{\partial \zeta} = -\frac{\sigma_0 a}{\sqrt{3}(\zeta a + s)} \tag{24}$$

and the elastic plastic boundary is determined with the use of (9) in the form

$$\zeta_1 = \left(e^{-k} - 4s\right) / (4a) \tag{25}$$

The magnitude of $\sigma_{\zeta\zeta}$ on the elastic side of the elastic plastic boundary $\zeta = \zeta_1$ can be found from (19) at $\zeta = \zeta_1$. Integrating (24) with the condition of continuity of $\sigma_{\zeta\zeta}$ across the elastic plastic boundary $\zeta = \zeta_1$ gives the distribution of the stress $\sigma_{\zeta\zeta}$ in the plastic zone $\zeta_1 \geq \zeta \geq -1$

$$\frac{\sigma_{\zeta\zeta}}{\sigma_0} = \frac{\sigma_{rr}}{\sigma_0} = \frac{1}{2\sqrt{3}k}\left\{\ln^2\left[4\left(a\zeta_1 + s\right)\right] - \ln^2\left[4\left(a\zeta_2 + s\right)\right]\right\} + \frac{1}{\sqrt{3}}\ln\left[\frac{\left(a\zeta_2 + s\right)\left(a\zeta_1 + s\right)}{s\left(a\zeta + s\right)}\right] \tag{26}$$

Using the identity $\sigma_{rr} - \sigma_{\theta\theta} \equiv s_{rr} - s_{\theta\theta}$, (10) and (26) leads to the distribution of the stress $\sigma_{\theta\theta}$ in the plastic zone $\zeta_1 \geq \zeta \geq -1$

$$\frac{\sigma_{\theta\theta}}{\sigma_0} = \frac{1}{2\sqrt{3}k}\left\{\ln^2\left[4\left(a\zeta_1 + s\right)\right] - \ln^2\left[4\left(a\zeta_2 + s\right)\right]\right\} + \frac{1}{\sqrt{3}}\ln\left[\frac{\left(a\zeta_2 + s\right)\left(a\zeta_1 + s\right)}{s\left(a\zeta + s\right)}\right] - \frac{2}{\sqrt{3}} \tag{27}$$

The pressure over the surface $r = r_{in}$ is determined from (26) at $\zeta = -1$

$$\frac{p}{\sigma_0} = -\frac{\sigma_{rr}\big|_{\zeta=-1}}{\sigma_0} = \frac{1}{2\sqrt{3}k}\left\{\ln^2\left[4\left(a\zeta_2 + s\right)\right] - \ln^2\left[4\left(a\zeta_1 + s\right)\right]\right\} + \frac{1}{\sqrt{3}}\ln\left[\frac{s\left(a\zeta + s\right)}{\left(a\zeta_2 + s\right)\left(a\zeta_1 + s\right)}\right] \tag{28}$$

4. BENDING MOMENT AND TENSILE FORCE

It follows from the equilibrium equations that the pressure over the surface $r = r_{in}$ is related to the tensile force per unit length T (Fig.1) by $T = pr_{in}$. Using (4) this equation can rewritten as

$$T = pH\sqrt{s-a}/a \tag{29}$$

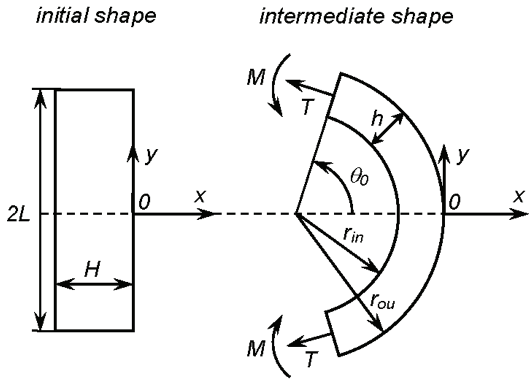

Fig. 1- Geometry of the process

Then, the average stress defined by $t = T/h$ follows from (4) and (29) in the form

$$t = \frac{p\sqrt{s-a}}{\sqrt{s}-\sqrt{s-a}} \tag{30}$$

It is seen from (29) and (30) that $T \to \infty$ and $t \to \infty$ as $a \to 0$, unless $p = O(a)$ or $p = o(a)$ as $a \to 0$. The pressure over the surface $r = r_{in}$ at the very beginning of the process is determined from (11) as

$$\frac{p}{\sigma_0} = -\frac{\sigma_{rr}|_{\zeta=-1}}{\sigma_0} = \frac{1}{2\sqrt{3}k}\left\{\ln^2\left(1-4a\zeta_0\right) - \ln^2\left[1-4a\left(1+\zeta_0\right)\right]\right\} \tag{31}$$

Expanding the right hand side of this equation in a series in the vicinity of $a = 0$ gives

$$\frac{p}{\sigma_0} = -\frac{8\left(1+2\zeta_0\right)}{\sqrt{3}k}a^2 + o\left(a^2\right) \qquad a \to 0 \tag{32}$$

Substituting (32) into (29) and (30) shows that $T = 0$ and $t = 0$ at $a = 0$ which is in agreement with physical expectations. The bending moment is defined by

$$M = \int_{r_{in}}^{r_{ou}} \left(\sigma_{\theta\theta} + p \right) r dr \tag{33}$$

Using (3) this equation can be rewritten in the form

$$M = \frac{H^2}{2a} \int_{-1}^{0} \left(\sigma_{\theta\theta} - t \right) d\zeta \tag{34}$$

Since the stress $\sigma_{\theta\theta}$ and t have been already found (equations (12), (20), (21), (27) and (30), the bending moment can be calculated numerically by means of (34) with no difficulty.

5. SUMMARY

An analytic ideal flow solution for non-steady flow of incompressible elastic perfectly plastic material has been obtained. The solution has been adopted to describe bending under tension of a wide sheet. It is expected that the solution found can be used for optimizing the process parameters to minimize springback. The solution can also been used to verify numerical codes for sheet metal forming.

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ELASTO – PLASTIČNO NESTACIONARNO REŠENJE TEČENJA MATERIJALA U IDEALNIM USLOVIMA I NJEGOVA PRIMENA U PROCESU SAVIJANJA UZ ISTOVREMENO ZATEZANJE

Elena Lyamina, Sergei Alexandrov

Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow, 119526, Russia

REZIME

U radu je dato originalno analitičko rešenje problema savijanja u uslovima ravanskog deformacionog stanja (savijanje uz istovremeno zatezanje) i to za slučaj velikih deformacija. Rešenje je bazirano na idealnoj teoriji tečenja za slučaj elastično – idealno plastično nekompresibilnih materijala. Realizovane su varijacije momenta savijanja i sile zatezanja.
Dobijena rešenja mogu se koristiti prilikom projektovanja procesa obrade lima, posebno uzimajući u obzir fenomen elastičnog vraćanja.
Ključne reči: *Savijanje u uslovima ravanskog deformacionog stanja, analitičko rešenje.*